
UNIT 3 CONTROL CHARTS FOR ATTRIBUTES

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3.1 INTRODUCTION

In Unit 2, you have learnt about control charts for variables, which are used to control the measurable quality characteristics. You have also learnt how to construct control charts for mean (\bar{X} -chart) and control chart for variability (R-chart and S-chart). However, there are many situations in which measurement is not possible, e.g., the number of failures in a production run, number of defects in a bolt of cloth, surface roughness of a cricket ball, etc. In such cases, we cannot use the control chart for variables. Moreover, control charts for variables can be used only for **one** measurable characteristic at a time. This may be problematic if a manufacturing company or a factory makes a product or a part having more than one measurable quality characteristic, say, 10 quality characteristics and the quality controller would like to control all the characteristics. Then for each quality characteristic, he/she would require a separate control chart for variables. However, it will be impracticable and uneconomical to have 10 such charts. In such situations, we use control charts for attributes, which provide overall information about quality.

You will learn about various aspects of control charts for attributes in this unit. In Sec. 3.2, we explain the meaning of the term **attributes** in industry and introduce different types of control charts for attributes. In Secs. 3.3 to 3.5, we discuss how to construct the control charts for attributes and interpret the result obtained from these charts for constant and variable sample size, respectively. In the next unit, we discuss the control charts for defects.

Objectives

After studying this unit, you should be able to:

- distinguish between the control charts for variables and attributes;
- explain the need for control charts for attributes;
- describe different types of control charts for attributes;
- decide on which control charts for attributes to use in a given situation;
- construct and interpret the control chart for fraction defective (p-chart);
- construct and interpret the p-chart for variable sample size; and
- construct and interpret the control chart for number of defectives (np-chart).

3.2 CONTROL CHARTS FOR ATTRIBUTES

We first define the term **attributes** as used in quality control terminology.

The term **attributes** in quality control refers to those quality characteristics, which classify the items/units into one of the two classes: conforming or non-conforming, defective or non-defective, good or bad.

There are two types of attributes:

- i) Where numerical measurements of the quality characteristics are not possible, for example, colour, scratches, damages, missing parts, etc.
- ii) Where numerical measurements of the quality characteristics are possible and items are classified as defective or non-defective on the basis of the inspection. For example, the diameter of a cricket ball can be measured by the micrometer but sometimes it may be more convenient to classify the balls as defective and non-defective using a Go-No-Go gauge (read the margin remark).

In inspection of attributes, actual measurements are not done, but the number of defective items (defectives) or number of defects in the item is counted. The size of defect and its location is not important. Items are inspected and either accepted or rejected.

There are different types of control charts for attributes for different situations. We classify the control charts for attributes into two groups as follows:

- Control charts for defectives, and
- Control charts for defects.

Control charts for defectives are mainly of two types as given below:

1. Control chart for fraction defective (p-chart), and
2. Control chart for number of defectives (np-chart).

Both control charts for defectives are based on the binomial distribution.

Control charts for defects are also of two types as given below:

1. Control chart for number of defects (c-chart), and
2. Control chart for number of defects per unit (u-chart).

Control charts for defects are based on Poisson distribution and these charts will be described in the next unit. In this unit, we focus on control **charts for defectives** and discuss the p-chart first.

3.3 CONTROL CHARTS FOR FRACTION DEFECTIVE (p-CHART)

The most widely used control chart for attributes is the **fraction (proportion) defective chart**, that is, the p-chart. The p-chart may be applied to quality characteristics, which cannot be measured or impracticable and uneconomical to measure it. These items/units are classified as defective or non-defective on the basis of certain criteria (defects). You have learnt that the control charts for variables are used to control only one quality characteristic at a time. If the quality controller would like to control two characteristics, he/she must use two

A Go-No-Go gauge is an inspection tool which is used to check an item or a unit or a piece against its allowed tolerances. The name **Go-No-Go** derives from its use. It means that we check the item and if the item is acceptable (fulfil the specifications), we say **Go** and if unacceptable, we say **No-Go**.

control charts for variables. But a single p-chart may be applied to more than one quality characteristic, in fact, to as many quality characteristics as we want.

Before describing the control chart for fraction defective (p-chart), we explain the meaning of **fraction defective**.

Fraction defective (p) is defined as the ratio of the number of defective terms/units/articles found in any inspection to the total number of items/units/articles inspected. Symbolically, we write

$\text{Fraction defective (p)} = \frac{\text{Number of defective items}}{\text{Total number of items inspected}} \quad \dots (1)$

For example, if 500 cricket balls are inspected and 20 balls are found defective, the fraction defective of the balls is given as:

$$\begin{aligned} \text{Fraction defective (p)} &= \frac{\text{Number of defective balls}}{\text{Total number of balls inspected}} \\ &= \frac{20}{500} = 0.04 \end{aligned}$$

Fraction defective is always less than or equal to 1 and expressed as a decimal or fraction.

We now explain how to construct the p-chart. The procedure of constructing a p-chart is similar to that of the control charts for variables. The main steps of constructing a p-chart are as follows:

Steps Involved in Construction of a p-chart

Step 1: We define non-conformity (defectiveness) properly because it depends on the product, its use, consumer needs, etc. For example, a scratch mark in a machine might not be considered non-conformity whereas the same scratch on a mobile phone, optical lens, laptop, etc. would be considered non-conformity.

Step 2: We determine the subgroup (sample) size and the number of subgroups (samples).

The selection of sample size for control charts for attributes is very important. The sample size for control charts for attributes should be large enough to allow non-conformities (non-conforming/defective items) to be observed in the samples. The sample size for control charts for attributes is a function of the proportion of non-conforming (defective) items. For example, if a process has 2% defective items, a sample of 20 items is not sufficient because the average number of defective items per sample is 0.40. It means that there is only 40% chance that defective items could be observed in the sample.

Therefore, misleading inference could be made. So for defective items, we would require many more samples. In such a situation, sample size of 100 should be sufficient. Thus, if a process is of very good quality, the p-chart requires large sample size to determine lack of control. If the process is of poor quality, small sample size is sufficient.

Generally, we decide the sample size as per the table given below:

Percentage of	Sample Size
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Defective	
0 to 10%	20
0 to 6%	100
1.06 to 2.94%	1000
1.81 to 2.19%	50000

Note that determining the sample size for a p-chart, we require some preliminary observation to obtain a rough idea about the proportion of defective items and their average number.

The number of samples depends upon the production rate and cost of sampling in addition to other factors. Generally, 25 samples are gathered to collect sufficient data.

Step 3: After deciding the size of the sample and the number of samples, we select sample units from the item produced randomly, so that each item has an equal chance of being selected.

Step 4: We collect the data for a p-chart in the same way as for the control chart for variables. However, instead of recording measurements of the items, we record the number of items inspected and the number of defective items.

Step 5: We calculate the fraction defective for each sample from equation (1).

If there are k samples of the same size n and d_1, d_2, \dots, d_k are the numbers of defectives in the 1st, 2nd, ..., kth sample, respectively, then for each sample, we calculate the fraction defective as follows:

$$p_1 = \frac{d_1}{n}, p_2 = \frac{d_2}{n}, \dots, p_k = \frac{d_k}{n} \quad \dots (2)$$

Step 6: We set the 3σ control limits to find out whether the process is under control or out-of-control with respect to the fraction defective. The 3σ control limits for the p-chart are given by:

$$\text{Centre line} = E(p) \quad \dots (3a)$$

$$\text{Upper control limit (UCL)} = E(p) + 3SE(p) \quad \dots (3b)$$

$$\text{Lower control limit (LCL)} = E(p) - 3SE(p) \quad \dots (3c)$$

For calculating the control limits of the p-chart, we need the sampling distribution of the fraction defectives.

You have learnt in Unit 2 of MST-004 that the mean and variance of the sampling distribution of the fraction (proportion) defective is given by

$$E(p) = P \text{ and } \text{Var}(p) = \frac{PQ}{n} \quad \dots (4a)$$

Here P is the probability or fraction or proportion of defective in the process and $Q = 1 - P$.

We also know the standard error of a random variable X is

$$SE(X) = \sqrt{\text{var}(X)}$$

Therefore, the standard error of sample fraction (proportion) defective is given by

$$SE(p) = \sqrt{\text{Var}(p)} = \sqrt{\frac{PQ}{n}} \quad \dots (4b)$$

You have learnt that the sampling distribution of the fraction defectives is not a normal distribution. However, if the sample size is sufficiently large, such that $nP > 5$ and $nQ > 5$, you know from the **centre limit theorem**, that the sampling distribution of the fraction defectives is approximately normally distributed with mean P and variance PQ/n . Therefore, the centre line and control limits for the p -chart are given as follows:

$$CL = E(p) = P \quad \dots (5a)$$

$$UCL = E(p) + 3SE(p) = P + 3\sqrt{\frac{P(1-P)}{n}} \quad \dots (5b)$$

$$LCL = E(p) - 3SE(p) = P - 3\sqrt{\frac{P(1-P)}{n}} \quad \dots (5c)$$

In practice, fraction defective (P) of the process is not known. So equations (5a to 5c) are not used to construct the control chart. It is necessary to estimate the fraction defective of the process. We estimate it using the sample fractions. The best estimator of the process fraction defective is the **average fraction defective** of the samples. Suppose, we draw k samples of the same size and for each sample, we calculate the fraction defective using equation (2). If p_1, p_2, \dots, p_k are the fraction defectives of the 1st, 2nd, ..., k^{th} sample, respectively, we calculate the **average fraction defective** as follows:

$$\bar{p} = \frac{1}{k}(p_1 + p_2 + \dots + p_k) = \frac{1}{k} \sum_{i=1}^k p_i \quad \dots (6)$$

Generally, this formula is used when we have sample fraction defectives. If the number of defective items for each sample is given, it is convenient to calculate the average fraction defective as follows:

$$\bar{p} = \frac{\text{Sum of defective items in all samples}}{\text{Total number of items inspected}} = \frac{1}{nk} \sum_{i=1}^k d_i \quad \dots (7)$$

In this case, the centre line and control limits of the p -chart are given as:

$$CL = E(p) = \hat{P} = \bar{p} \quad \dots (8a)$$

$$UCL = E(p) + 3SE(p) = \hat{P} + 3\sqrt{\frac{\hat{P}(1-\hat{P})}{n}} = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} \quad \dots (8b)$$

$$LCL = E(p) - 3SE(p) = \hat{P} - 3\sqrt{\frac{\hat{P}(1-\hat{P})}{n}} = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} \quad \dots (8c)$$

You have studied the **Central Limit Theorem** in Unit1 of MST-004.

Step 7: Having set the centre line and control limits, we construct the p-chart by taking the sample number on the X-axis and the sample fraction defective (p) on the Y-axis. We draw the control line as a solid line, and the UCL and LCL as dotted lines. Then we plot the value of the sample fraction defective against the sample number. The consecutive sample points are joined by line segments.

Step 8: We interpret the result. If all sample points lie on or in between the upper and lower control limits, the control chart indicates that the process is under control. That is, only chance causes are present in the process. If one or more points lie outside the upper or lower control limits, the control chart alarms (indicates) that the process is not under statistical control and some assignable causes are present in the process.

To bring the process under statistical control, it is necessary to investigate the assignable causes and take corrective action to eliminate them. Once the assignable causes are eliminated, we delete the out-of-control points (samples) and calculate the **revised centre line** and **control limits** for the p-chart by using the remaining samples. These limits are known as the **revised control limits**. To calculate the revised limits of the p-chart, we first calculate new \bar{p} as follows:

$$\bar{p}_{new} = \frac{\sum_{i=1}^k p_i - \sum_{j=1}^d p_j}{k-d} \quad \text{or} \quad \bar{p}_{new} = \frac{\sum_{i=1}^k d_i - \sum_{j=1}^d d_j}{n(k-d)} \quad \dots (9)$$

where d – the number of discarded samples,

$\sum_{j=1}^d p_j$ – the sum of fraction defectives in the discarded samples, and

$\sum_{j=1}^d d_j$ – the sum of number of defects in the discarded samples.

After finding \bar{p}_{new} , we reconstruct the centre line and control limits of the p-chart by replacing \bar{p} by \bar{p}_{new} in equations (8a to 8c) as follows:

$CL = \bar{p}_{new}$... (10a)
$UCL = \bar{p}_{new} + 3\sqrt{\frac{\bar{p}_{new}(1-\bar{p}_{new})}{n}}$... (10b)
$LCL = \bar{p}_{new} - 3\sqrt{\frac{\bar{p}_{new}(1-\bar{p}_{new})}{n}}$... (10c)

Note 1: If the value of lower control limit is negative, we change the lower control limit to zero because a negative fraction defective is impossible.

Note 2: If the p-chart continuously indicates an increase in the value of the average fraction (proportion) defective, the management should investigate the reasons behind this increase rather than constantly revising the centre line and

control limits. The possible reasons behind the increase in the defective items may be poor incoming quality from vendors, tightening of specification limits, etc.

So far, we have discussed various steps involved in the construction of the p-chart. Let us now take up some examples to illustrate this method.

Example 1: In a company manufacturing cricket ball, the quality controller inspects the balls and classifies them as defective or non-defective on the basis of certain defects. The company manager wants to maintain the process so that an average of not more than 5 percent of the output is defective. Suggest a suitable control chart for this purpose. If the company can work with a sample of size 500, calculate the centre line and control limits for this chart.

Solution: Here the balls are classified as defective or non-defective. So we use the control chart for attributes. The quality controller has to maintain the process so that an average of not more than 5% of the output is defective. It means that the process proportion defective is given. So we use the p-chart.

We are given that $P = 5\% = 0.05$ and $n = 500$.

Therefore, we can calculate the centre line and control limits from equations (5a to 5c) as follows:

$$CL = P = 0.05$$

$$\begin{aligned} UCL &= P + 3\sqrt{\frac{P(1-P)}{n}} = 0.05 + 3\sqrt{\frac{0.05 \times (1-0.05)}{500}} \\ &= 0.05 + 3\sqrt{0.000095} = 0.0792 \end{aligned}$$

$$\begin{aligned} LCL &= P - 3\sqrt{\frac{P(1-P)}{n}} = 0.05 - 3\sqrt{\frac{0.05 \times (1-0.05)}{500}} \\ &= 0.05 - 3\sqrt{0.000095} = 0.0208 \end{aligned}$$

Example 2: A factory manufacturing small bolts. To check the quality of the bolts, the manufacturer selected 20 samples of same size 100 from the manufacturing process time to time. He/she visually inspected each selected bolt for certain defects. After the inspection, he/she obtained the following data:

Sample Number	1	2	3	4	5	6	7	8	9	10
Proportion Defective	0.10	0.04	0.08	0.15	0.08	0	0.01	0.05	0.05	0.08
Sample Number	11	12	13	14	15	16	17	18	19	20
Proportion Defective	0.10	0	0.06	0.05	0.03	0.20	0.05	0.07	0.01	0.08

Estimate the proportion defective of the process. Does the process appear to be under control with respect to the proportion of defective bolts?

Solution: We know that the proportion defective of the process can be estimated by average proportion defective of the samples. We can calculate it using equation (6) as follows:

$$\hat{P} = \bar{p} = \frac{1}{k} \sum p = \frac{1}{20} \times 1.29 = 0.065$$

Process Control

Here the manufacturer visually inspected each selected bolt and classified it as defective or non-defective. So we use the control chart for attributes. Since the proportion defective for each sample is given, we can use the p-chart.

Here the process proportion defective is not known. So we use equations (8a to 8c) to obtain the centre line and control limits for the p-chart as follows:

$$CL = \bar{p} = 0.065$$

$$UCL = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} = 0.065 + 3\sqrt{\frac{0.065 \times (1-0.065)}{100}}$$

$$= 0.065 + 3 \times 0.025 = 0.140$$

$$LCL = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} = 0.065 - 3\sqrt{\frac{0.065 \times (1-0.065)}{100}}$$

$$= 0.065 - 3 \times 0.025 = -0.01 \square 0$$

We now construct the p-chart. We draw the centre line as a solid line and control limits as dotted lines on the graph and plot the points by taking the sample number on the X-axis and the proportion defective (p) on the Y-axis as shown in Fig. 3.1.

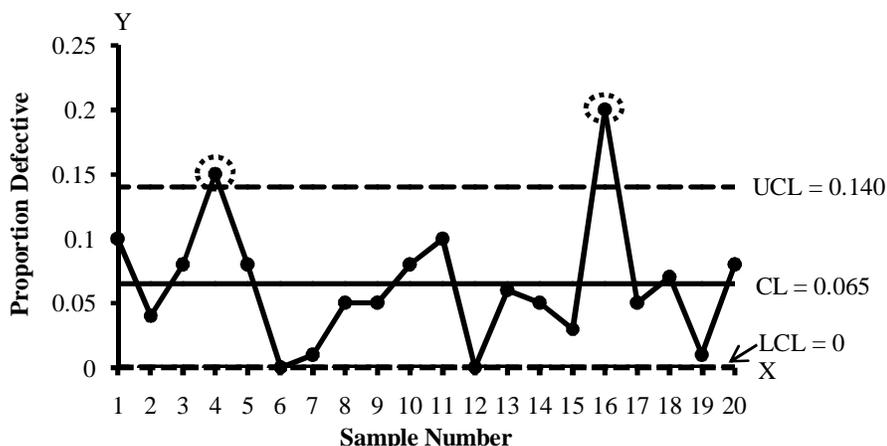


Fig. 3.1: The p-chart for proportion of defective bolts.

Interpretation of the result

From the p-chart shown in Fig. 3.1, we observe that the points corresponding to sample numbers 4 and 16 lie outside the control limits. So we can say that the process is not under statistical control with respect to the proportion of defective bolts.

Example 3: The following data are found during the inspection of the first 15 samples of size 100 each from a lot of two-wheelers manufactured by an automobile company:

Sample Number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Number of Defectives	3	4	6	2	12	5	3	6	3	5	4	15	5	2	3

Draw the chart for fraction defective (p) and comment on the state of control. If the process is out-of-control, calculate the revised centre line and control limits by assuming assignable causes for any out-of-control point.

Solution: To draw the p-chart, we need to calculate the centre line and control limits. Here, the process fraction defective is not known. So in this case we use equations (8a to 8c).

To calculate the control limits, we first calculate the fraction defective for each sample and then \bar{p} .

Sample Number	Sample Size (n)	Number of Defectives (d)	Proportion Defective (p = d/n)
1	100	3	0.03
2	100	4	0.04
3	100	6	0.06
4	100	2	0.02
5	100	12	0.12
6	100	5	0.05
7	100	3	0.03
8	100	6	0.06
9	100	3	0.03
10	100	5	0.05
11	100	4	0.04
12	100	15	0.15
13	100	5	0.05
14	100	2	0.02
15	100	3	0.03
Total		78	

$$\text{Proportion defective } p = \frac{\text{Number of defectives}}{\text{Sample size}}$$

From the above table, we have

$$\bar{p} = \frac{1}{nk} \sum d = \frac{1}{15 \times 100} \times 78 = 0.052$$

Therefore, we can calculate the centre line and control limits as follows:

$$CL = \bar{p} = 0.052$$

$$UCL = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} = 0.052 + 3\sqrt{\frac{0.052 \times (1-0.052)}{100}}$$

$$= 0.052 + 3 \times 0.022 = 0.118$$

$$LCL = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} = 0.052 - 3\sqrt{\frac{0.052 \times (1-0.052)}{100}}$$

$$= 0.052 - 3 \times 0.022 = -0.014 \square 0$$

We now draw the p-chart by taking the sample number on the X-axis and the proportion defective (p) on the Y-axis as shown in Fig. 3.2.

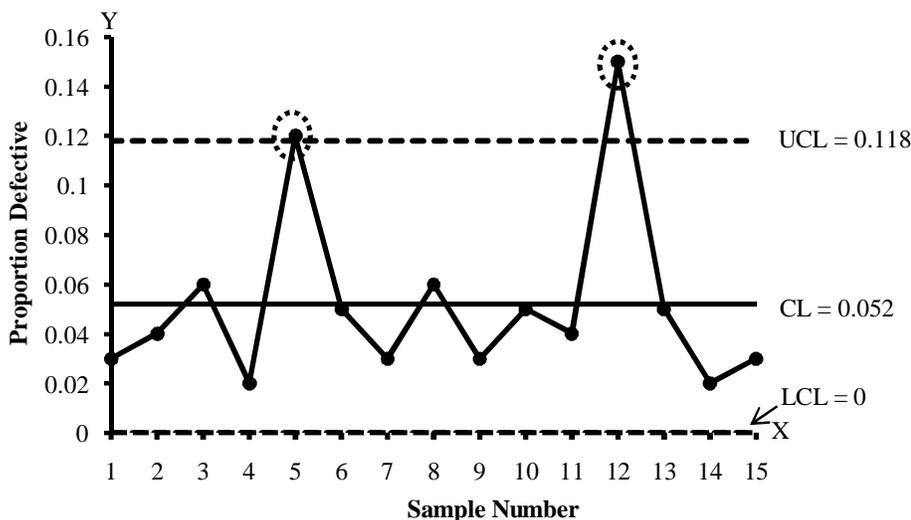


Fig. 3.2: The p-chart for fraction defective of two-wheelers.

Interpretation of the result

From the p-chart shown in Fig. 3.2., we observe that the points corresponding to sample numbers 5 and 12 lie outside the upper control limits. Therefore, the process is out-of-control. It means that some assignable causes are present in the process.

For calculating the revised limits for the p-chart, we first delete the out-of-control points (5 and 12) and calculate the new \bar{p} using remaining samples.

In our case $n = 100$, $k = 15$, $d = 2$, $\sum_{j=1}^2 d_j = 12 + 15 = 27$

$$\therefore \bar{p}_{new} = \frac{\sum_{i=1}^k d_i - \sum_{j=1}^d d_j}{n(k-d)} = \frac{78 - 27}{100(20-2)} = \frac{51}{1800} = 0.028$$

After finding the \bar{p}_{new} , we reconstruct the centre line and control limits of the chart using equations (10a to 10c) as follows:

$$CL = \bar{p}_{new} = 0.028$$

$$UCL = \bar{p}_{new} + 3\sqrt{\frac{\bar{p}_{new}(1-\bar{p}_{new})}{n}} = 0.028 + 3\sqrt{\frac{0.028 \times (1-0.028)}{100}}$$

$$= 0.028 + 3 \times 0.016 = 0.0716$$

$$LCL = \bar{p}_{new} - 3\sqrt{\frac{\bar{p}_{new}(1-\bar{p}_{new})}{n}} = 0.028 - 3\sqrt{\frac{0.028 \times (1-0.028)}{100}}$$

$$= 0.028 - 3 \times 0.016 = -0.020 \square 0$$

You may like to pause here and check your understanding of the p-chart by answering the following exercises.

E1) Choose the correct option from the following:

- i) The control chart used for attributes is the
 - a) p-chart b) np-chart c) c-chart d) all a), b) and c)

- ii) The control chart for proportion defective is the
 a) \bar{X} -chart b) p-chart c) np-chart d) c-chart
- iii) The upper control limit for the p-chart is
 a) $\bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$ b) $\bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$
 c) $\bar{p} - 3\sqrt{\bar{p}(1-\bar{p})}$ d) $\bar{p} + 3\sqrt{\bar{p}(1-\bar{p})}$
- iv) If the lower control limit has a negative value in a p-chart, it is treated as
 a) zero b) unity c) negative only d) positive only

E2) List the different types of control charts for attributes.

E3) Write the control limits of the p-chart when the fraction defective of the process is not known.

E4) A daily sample of 30 shirts was taken over a period of 15 days in order to monitor the manufacturing process of the shirts. Each shirt was inspected and classified as defective or non-defective. If a total of 22 defective shirts were found in 15 days, what should be the upper and lower control limits of the proportion of defective shirts?

E5) A mobile manufacturer inspects 30 mobiles at the end of the day of production and notes the number of defective mobiles. This procedure is continued up to 12 days and 2, 1, 3, 0, 2, 1, 0, 5, 2, 0, 3, 1 defective mobile are found. Is the production process under control with respect to the proportion defective?

3.4 p-CHART FOR VARIABLE SAMPLE SIZE

In Sec. 3.3, we have discussed the p-chart for constant sample size, that is, we have inspected the same number of items in each sample and the number of defective items has been counted. Sometimes, situations arise wherein we cannot take the same number of items in each sample. Generally, this happens when the p-chart is used for 100% inspection of output that varies from day to day. The variation in the output per day can be due to breakdown of machines, different production requirements, etc. When the sample size is not uniform (same or constant), we use the p-chart for variable sample size.

The procedure of constructing the p-chart for variable sample size is similar to the p-chart with uniform (constant) sample size except for the control limits.

Since the control limits are function of sample size (n), these will vary with the sample size. So for the p-chart for variable sample size, the control limits are calculated separately for each sample. The control limits in such a situation are known as **variable control limits**.

Generally, two approaches are used for constructing variable control limits for the p-chart.

First Approach

Process Control

According to the first approach, we calculate the control limits for each sample. Let n_i be the size of the i^{th} sample. The centre line and control limits of the i^{th} sample for the p-chart when the standard value of fraction defective P is known are given as follows:

$$CL = P \quad \dots (11a)$$

$$UCL = P + 3 \sqrt{\frac{P(1-P)}{n_i}} \quad \dots (11b)$$

$$LCL = P - 3 \sqrt{\frac{P(1-P)}{n_i}} \quad \dots (11c)$$

When P is unknown, it is estimated by the sample average fraction (proportion) defective, i.e., \bar{p} as we have discussed in Sec. 3.3. That means, we first calculate the fraction defective for each sample using equation (1).

If d_1, d_2, \dots, d_k represent the number of defectives in $1^{\text{st}}, 2^{\text{nd}}, \dots, k^{\text{th}}$ samples of size n_1, n_2, \dots, n_k , respectively, then

$$p_1 = \frac{d_1}{n_1}, p_2 = \frac{d_2}{n_2}, \dots, p_k = \frac{d_k}{n_k} \quad \dots (12a)$$

and we calculate \bar{p} as follows:

$$\bar{p} = \frac{d_1 + d_2 + \dots + d_k}{n_1 + n_2 + \dots + n_k} = \frac{\sum_{i=1}^k d_i}{\sum_{i=1}^k n_i} \quad \text{or} \quad \bar{p} = \frac{1}{k} \sum_{i=1}^k p_i \quad \dots (12b)$$

The centre line remains constant for each sample and is given by

$$CL = \bar{p} \quad \dots (13a)$$

The control limits change due to variable sample size. The control limits for the i^{th} sample are given by

$$UCL_i = \hat{P} + 3 \sqrt{\frac{\hat{P}(1-\hat{P})}{n_i}} = \bar{p} + 3 \sqrt{\frac{\bar{p}(1-\bar{p})}{n_i}} \quad \dots (13b)$$

$$LCL_i = \hat{P} - 3 \sqrt{\frac{\hat{P}(1-\hat{P})}{n_i}} = \bar{p} - 3 \sqrt{\frac{\bar{p}(1-\bar{p})}{n_i}} \quad \dots (13c)$$

Second Approach

According to the second approach, the control limits are calculated using the average sample size. This approach is used only when there is no large variation in the sample sizes and it is expected that the future sample sizes will not differ significantly from the average sample size. Using this approach, we get constant control limits just as we get in the case of constant sample size.

We calculate the average sample size as follows:

$$\bar{n} = \frac{n_1 + n_2 + \dots + n_k}{k} = \frac{1}{k} \sum_{i=1}^k n_i \quad \dots (14)$$

The centre line and control limits when P is known are given as follows:

$CL = P$... (15a)
$UCL = P + 3\sqrt{\frac{P(1-P)}{\bar{n}}}$... (15b)
$LCL = P - 3\sqrt{\frac{P(1-P)}{\bar{n}}}$... (15c)

However, when P is not known, the centre line and control limits are given as

$CL = \bar{p}$... (16a)
$UCL = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{\bar{n}}}$... (16b)
$LCL = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{\bar{n}}}$... (16c)

follows:

Note: The p-chart for variable sample size should only be used when samples of constant size are not possible, unavailable or unrealistic.

Let us explain how to construct this chart.

Example 4: In the production of tyres of an automobile, the output of a given size was inspected every day prior to the tyres being given to finished goods stores. The number of defective tyres found every day inspection is summarised in the following table:

Number of Sample	Number of Tyres Inspected	Number of Defective Tyres	Number of Sample	Number of Tyres Inspected	Number of Defective Tyres
1	650	70	11	670	71
2	510	74	12	660	75
3	600	58	13	600	77
4	590	61	14	550	78
5	630	65	15	540	64
6	650	115	16	610	90
7	700	82	17	670	96
8	740	55	18	660	110
9	580	80	19	650	78
10	600	90	20	590	60

Draw the appropriate control chart and comment on the state of the process.

Solution: Since the number of defectives is given and sample size varies, we use the p-chart for variable sample size. We shall solve this problem using both approaches described in this section.

First Approach

Process Control

According to the first approach, the control limits of the p-chart are calculated for each sample.

Since the average fraction defective of the process is not given, we use equations (13a to 13c) to calculate the centre line and control limits of the 1th sample.

For this, we first calculate the proportion of defective tyres (p) for each sample and \bar{p} using equations (12a to 12b).

Number of Sample	Number of Tyres Inspected (n)	Number of Defective Tyres (d)	Proportion of Defective Tyres (p = d/n)	CL	UCL	LCL
1	650	70	0.108	0.124	0.163	0.085
2	510	74	0.145	0.124	0.168	0.080
3	600	58	0.097	0.124	0.164	0.084
4	590	61	0.103	0.124	0.165	0.083
5	630	65	0.103	0.124	0.163	0.085
6	650	115	0.177	0.124	0.163	0.085
7	700	82	0.117	0.124	0.161	0.087
8	740	55	0.074	0.124	0.160	0.088
9	580	80	0.138	0.124	0.165	0.083
10	600	90	0.150	0.124	0.164	0.084
11	670	71	0.106	0.124	0.162	0.086
12	660	75	0.114	0.124	0.162	0.086
13	600	77	0.128	0.124	0.164	0.084
14	550	78	0.142	0.124	0.166	0.082
15	540	64	0.119	0.124	0.167	0.081
16	610	90	0.148	0.124	0.164	0.084
17	670	96	0.143	0.124	0.162	0.086
18	660	110	0.167	0.124	0.162	0.086
19	650	78	0.120	0.124	0.163	0.085
20	590	60	0.102	0.124	0.165	0.083
Total	12450	1549				

From the above table, we have

$$\bar{p} = \frac{\sum_{i=1}^k d_i}{\sum_{i=1}^k n_i} = \frac{1549}{12450} = 0.124$$

The centre line and control limits (shown in the above table) are calculated using equations (13a to 13c) as follows:

$$CL = \bar{p} = 0.124$$

$$\begin{aligned} UCL_1 &= \bar{p} + 3 \sqrt{\frac{\bar{p}(1-\bar{p})}{n_1}} = 0.124 + 3 \sqrt{\frac{0.124 \times (1-0.124)}{650}} \\ &= 0.124 + 3 \times 0.013 = 0.163, \text{ and so on.} \end{aligned}$$

$$\begin{aligned} LCL_1 &= \bar{p} - 3 \sqrt{\frac{\bar{p}(1-\bar{p})}{n_1}} = 0.124 - 3 \sqrt{\frac{0.124 \times (1-0.124)}{650}} \\ &= 0.124 - 3 \times 0.013 = 0.085, \text{ and so on.} \end{aligned}$$

We draw the p-chart by taking the sample number on the X-axis and the proportion defective (p) of tyres on the Y-axis as shown in Fig. 3.3.

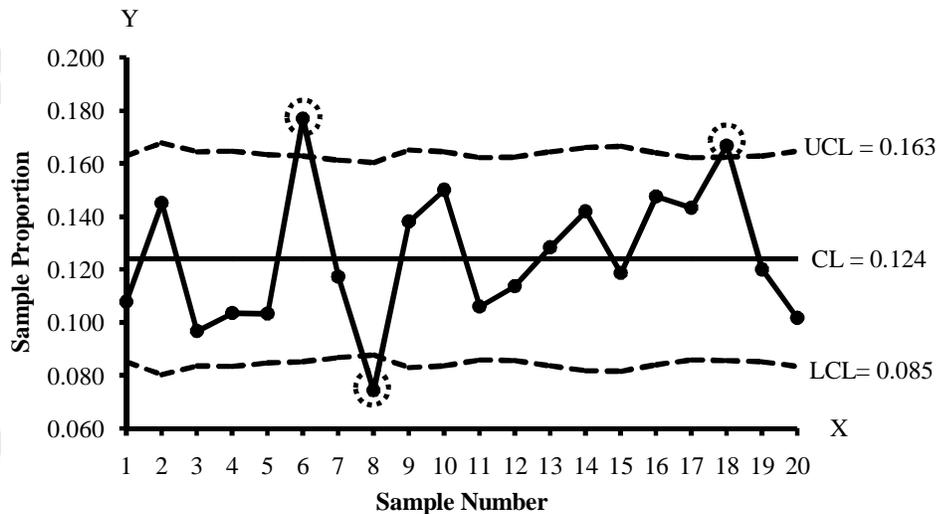


Fig. 3.3: The p-chart for proportion of defective tyres.

Interpretation of the result

From the p-chart shown in Fig. 3.3, we observe that the points corresponding to sample numbers 6, 8 and 18 lie outside the control limits. Therefore, we can say that the process is out-of-control with respect to the proportion of defective tyres and some assignable causes are present in it.

Let us now solve this example by using the second approach.

Second Approach

According to this approach, the control limits are calculated using the average sample size and we get constant control limits just as we get in the case of constant sample size. We can calculate the average sample size using equation (14) as follows:

$$\bar{n} = \frac{1}{k} \sum_{i=1}^k n_i = \frac{1}{20} \times 12450 = 622.5$$

Using equations (16a to 16c), we calculate the centre line and control limits as follows:

$$CL = \bar{p} = 0.124$$

$$\begin{aligned} UCL &= \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{\bar{n}}} = 0.124 + 3\sqrt{\frac{0.124 \times (1-0.124)}{622.5}} \\ &= 0.124 + 3 \times 0.013 = 0.163 \end{aligned}$$

$$\begin{aligned} LCL &= \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{\bar{n}}} = 0.124 - 3\sqrt{\frac{0.124 \times (1-0.124)}{622.5}} \\ &= 0.124 - 3 \times 0.013 = 0.085 \end{aligned}$$

Then, we plot the p-chart (see Fig. 3.4).

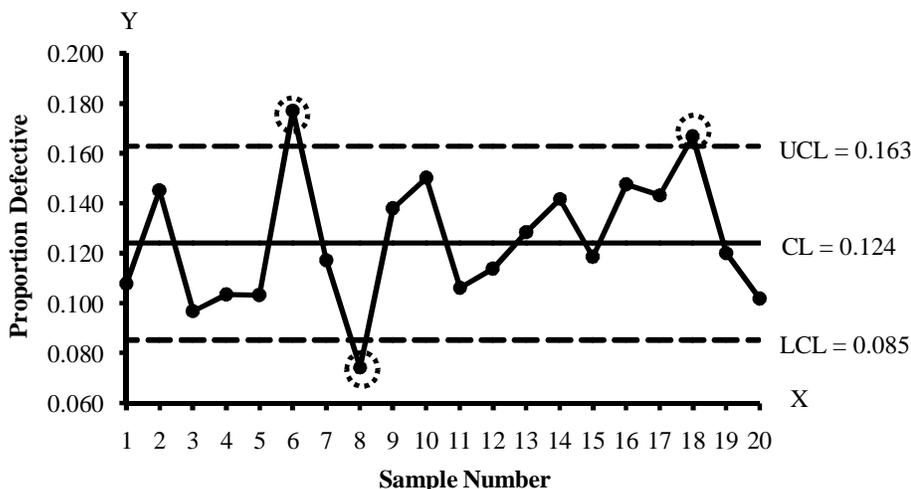


Fig. 3.4: The p-chart for proportion of defective tyres.

Interpretation of the result

From Fig. 3.4, we note that the points corresponding to sample numbers 6, 8 and 18 lie outside the control limits. Therefore, the process is out-of-control.

Hence, both approaches give the same conclusion about the process.

Now, you can try the following exercise and draw the p-chart for variable sample size.

E6) To monitor the manufacturing process of laptops, a quality control engineer randomly selects a number of laptops from the production line, each day over a period of 20 days. The laptops are inspected for certain defects and the number of defective laptops found each day is recorded in the following table:

Day	Number of Laptops Inspected	Number of Defective Laptops	Day	Number of Laptops Inspected	Number of Defective Laptops
1	50	4	11	55	6
2	52	5	12	60	1
3	57	6	13	55	5
4	50	10	14	55	8
5	50	4	15	52	2
6	48	3	16	48	2
7	51	4	17	50	7
8	54	7	18	56	9
9	52	8	19	52	2
10	50	4	20	53	4

Construct the p-chart for variable sample size with the help of both approaches and comment on the process.

3.5 CONTROL CHART FOR NUMBER OF DEFECTIVES (np-CHART)

In Sec. 3.4, you have learnt that for the p-chart, the fraction (proportion) defective for each sample is calculated and plotted against the sample number. Alternatively, instead of calculating the proportion defective, we can **count** the number of defective items in the samples and plot them against the sample number. The chart thus obtained is known as the np-chart.

The np-chart is an adaptation of the basic p-chart. The purpose of this chart is to provide a better understanding to the operator who may understand actual number of defective items easily rather than the proportion defective. However, there is one drawback of the np-chart: It is used only for equal sample size because the number of defectives cannot be compared for variable sample size. For example, suppose we get one defective item in a sample of size 10 and one defective item in a sample of size 100. In both samples, we get one defective item, but the first sample shows a much higher rate of problems, that is 10%, in comparison to the second.

The name np is due to the reason that the proportion defective p was obtained by dividing the actual number of defectives (d) by the sample size (n), i.e., $p = d/n$. So $d = np$. Therefore, the actual number of defectives may be represented by np.

The underlying logic and basic form of the np-chart are similar to the p-chart. The primary difference between the np-chart and p-chart is that instead of plotting the sample fraction defectives and monitoring their variation, we plot the number of defectives per sample and monitor them. Therefore, in this section, we describe the centre line and control limits of the np-chart.

For obtaining the control limits of the np-chart, we need the sampling distribution of the number of defectives (d). Recall the sampling distribution of the number of defectives described in Unit 2 of MST-004. Suppose the items or units of a process (population) are classified into two mutually exclusive groups as defective (success) and non-defective (failure) and a random sample of size n is drawn from it. Then the number of defectives follows a binomial distribution with mean nP and variance nPQ where P is the fraction or proportion of defectives in the process (population) and $Q = 1 - P$:

$$E(d) = nP \text{ and } \text{Var}(d) = nP(1 - P) \quad \dots (17)$$

We also know that the standard error of a random variable (X) is

$$SE(X) = \sqrt{\text{var}(X)}$$

Therefore, the standard error of the number of defectives is given by

$$SE(d) = \sqrt{\text{Var}(d)} = \sqrt{nP(1 - P)} \quad \dots (18)$$

The sampling distribution of number of defectives is not a normal distribution. However, if the sample size is sufficiently large, you know from the **centre limit theorem** that the sampling distribution of number of defectives is approximately normally distributed with mean nP and variance nPQ. Therefore, the centre line and control limits for the np-chart are given as follows:

$$\text{Centre line (CL)} = E(d) = nP \quad \dots (19a)$$

$$\text{Upper control limit (UCL)} = E(d) + 3SE(d) = nP + 3\sqrt{nP(1 - P)} \quad \dots (19b)$$

$$\text{Lower control limit (LCL)} = E(d) - 3SE(d) = nP - 3\sqrt{nP(1 - P)} \quad \dots (19c)$$

np-chart is also known as d-chart.

Process Control

In practice, the fraction (proportion) defective (P) of the process is not known. Therefore, it is necessary to estimate it by the average sample fraction defective as we have discussed in Sce.3.3.

If we draw k samples of same size n and d_1, d_2, \dots, d_k are the numbers of defective items in 1st, 2nd, ..., k^{th} sample, respectively, we can calculate the average fraction defective of the samples as follows:

$$\bar{p} = \frac{\text{Sum of defective items}}{\text{Total number of items inspected}} = \frac{1}{nk} \sum_{i=1}^k d_i \quad \dots (20)$$

Hence, in this case, the centre line and control limits of the np-chart are obtained by replacing P by \bar{p} in equations (19a to 19c) as follows:

$$CL = E(d) = n\hat{P} = n\bar{p} \quad \dots (21a)$$

$$UCL = E(d) + 3SE(d) = n\hat{P} + 3\sqrt{n\hat{P}(1-\hat{P})} = n\bar{p} + 3\sqrt{n\bar{p}(1-\bar{p})} \quad \dots (21b)$$

$$LCL = E(d) - 3SE(d) = n\hat{P} - 3\sqrt{n\hat{P}(1-\hat{P})} = n\bar{p} - 3\sqrt{n\bar{p}(1-\bar{p})} \quad \dots (21c)$$

Having set the centre line and control limits, we construct the np-chart by taking the sample number on the X-axis and the number of defective items (d) in the sample on the Y-axis. We draw the control line as a solid line and UCL and LCL as dotted lines. Then we plot the number of defective items for each sample against the sample number. The consecutive sample points are joined by line segments.

Interpretation of the result

If all sample points lie on or in between upper and lower control limits, the control chart indicates that the process is under statistical control, that is, only chance causes are present in the process. If one or more points lie outside the upper or lower control limits, the control chart alarms (indicates) that the process is not under statistical control and some assignable causes are present in the process.

To bring the process under statistical control it is necessary to investigate the assignable causes and take corrective action to eliminate them. Once the assignable causes are eliminated, we delete the out-of-control points and calculate the revised centre line and control limits for the np-chart by using the remaining samples. These limits are known as the **revised control limits**. To obtain the revised limits for the np-chart, we first calculate the new \bar{p} as follows:

$$\bar{p}_{\text{new}} = \frac{\sum_{i=1}^k d_i - \sum_{j=1}^d d_j}{n(k-d)} \quad \dots (22)$$

where d – the number of discarded samples, and

$\sum_{j=1}^d d_j$ – the sum of number of defectives in the discarded samples.

We then reconstruct the centre line and control limits of the chart by replacing \bar{p} by \bar{p}_{new} in equations (21 to 21c) as follows:

$CL = n\bar{p}_{new}$... (23a)
$UCL = n\bar{p}_{new} + 3\sqrt{n\bar{p}_{new}(1-\bar{p}_{new})}$... (23b)
$UCL = n\bar{p}_{new} - 3\sqrt{n\bar{p}_{new}(1-\bar{p}_{new})}$ (23c)	...

Let us explain how to draw the np-chart with the help of examples.

Example 5: To monitor the manufacturing process of laptops, a quality control engineer randomly selects 50 laptops from the production line, each day over a period of 20 days. The laptops are inspected for certain defects and the number of defective laptops found each day is recorded in the following table:

Day	Number of Laptops Inspected	Number of Defective Laptops	Day	Number of Laptops Inspected	Number of Defective Laptops
1	50	4	11	50	6
2	50	8	12	50	1
3	50	6	13	50	5
4	50	10	14	50	3
5	50	4	15	50	2
6	50	3	16	50	3
7	50	4	17	50	7
8	50	7	18	50	9
9	50	8	19	50	2
10	50	4	20	50	4

Construct the appropriate control chart and state whether the process is in control.

Solution: In this case, the quality control engineer visually inspects each selected laptop to determine whether it is defective or not. So we use the control chart for attributes. Since the number of defectives is given, we can use p or np-chart. But it is convenient to use the np-chart.

We do not know the process fraction defective. Therefore, we use equations (21a to 21c) to calculate the centre and control limits for the np-chart.

We are given that

the number of laptops inspected in each sample (n) = 50 and

the number of samples (k) = 20.

The total number of defectives is $\sum_{i=1}^k d_i = 4 + 8 + \dots + 2 + 4 = 100$.

We calculate \bar{p} from equation (20) as follows:

$$\bar{p} = \frac{1}{nk} \sum_{i=1}^k d_i = \frac{1}{50 \times 20} \times 100 = 0.10$$